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Effect of boundary radiation on thermal stability in horizontal layers

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Abstract—Thermal radiation across radiatively non-participating horizontal fluid layers increases the critical Rayleigh number by affecting the thermal boundary conditions. Modification of the convective boundary condition by radiation can be characterized by two modified Biot numbers that include a radiative contribution. Disturbance amplitude ratios needed to calculate the modified Biot numbers are presented. The wave number of the most unstable disturbance is shown to depend primarily on critical Rayleigh number and only weakly on the ratio of the modified Biot numbers, thus facilitating engineering applications.

INTRODUCTION

SOME TIME ago Sparrow *et al.* [1] showed that convective boundary conditions affect thermal stability in the horizontal layer heated from below. A disturbance in the layer is damped by thermal dissipation as well as viscous dissipation, as is shown by the presence of thermal conductivity in the Rayleigh number [2]. Less than perfectly conducting horizontal boundaries do not damp thermal perturbations as strongly in the vicinity of the walls, and this effect can lower the critical Rayleigh number from 1708 for perfectly conducting walls to 720 for poorly conducting walls. Radiation transfer from walls through a diathermanous gas has been shown to increase thermal stability in honeycombs [3], and a similar effect is expected for horizontal layers.

The effect of thermal boundary conditions on multiple layers has been addressed in a series of papers by Catton and Lienhard [4–6]. Here too the presence of a less than perfectly conducting solid midlayer between fluid layers has the effect of lowering critical Rayleigh number. However, when the midlayer exchanges heat by radiation with the upper and lower boundaries, the radiation acts to damp thermal perturbations and tends to stabilize the fluid layers.

In what follows the radiative interaction for the single horizontal layer and two symmetric horizontal layers separated by a thin dielectric film is developed. It is shown that the radiative interaction depends upon the wave number of the disturbance and the ratio of temperature perturbation amplitudes on the horizontal boundaries. The needed amplitudes are calculated in the manner of Sparrow *et al.* [1], and the wave number of the most unstable disturbance is found as a function of the governing dimensionless parameters.

THEORY

The single layer

The linearized time-independent perturbation equations governing the non-dimensional temperature disturbance θ^* and non-dimensional z-velocity component W^* are [2]

$$(D_z^2-a^2)\theta^* = -Pr\, W^* \tag{1}$$

$$(D_z^2-a^2)W^* = a^2\, Gr\, \theta^* \tag{2}$$

$$(D_z^2-a^2)^3\theta^* = -a^2\, Ra\, \theta^*. \tag{3}$$

In the problem treated by Sparrow *et al.* the boundary conditions are zero velocity, zero first derivative of W^* (from continuity), and continuity in heat flux and temperature at the upper and lower boundaries

$$W^* = 0, \quad z^* = 0, 1 \tag{4}$$

$$D_z W^* = 0, \quad z^* = 0, 1 \tag{5}$$

$$D_z\theta^* = Bi_0\theta^*, \qquad z^* = 0 \tag{6a}$$

$$D_z\theta^* = -Bi_1\theta^*, \quad z^* = 1 \tag{6b}$$

where the Biot numbers at the lower and upper boundaries are

$$Bi_0 = \frac{h_{c,0}L}{K}, \quad Bi_1 = \frac{h_{c,1}L}{K}. \tag{7a, b}$$

When the fluid is transparent, radiation can no longer be considered a contributor to heat conduction in the fluid through the Rosseland diffusion equation, but must be considered separately. Thus the heat flux balance at the lower wall (in dimensional variables) has a radiant flux q_0 in addition to the fluid conduction and boundary conduction terms, as shown in Fig. 1

$$h_{c,0}(T_{0,e}-T(x,0)) = -K\frac{\partial T}{\partial z}\bigg|_{z=0} + q_0. \tag{8}$$

NOMENCLATURE

a	dimensionless perturbation wave number
B	black body radiosity
Bi	Biot number
c	speed of light
D_z	dimensionless derivative with respect to z ($L\,d/dz$)
F	transfer factor
F_s	shape factor
Gr	Grashof number
h	Planck constant
h_c, h_r	heat transfer coefficients
K	fluid conductivity
k	Boltzmann constant
L	fluid layer thickness
Pr	Prandtl number
q	radiant heat flux
R_θ	ratio of the temperature perturbation amplitudes at the upper and lower boundaries, equation (24)
Ra	Rayleigh number
$S(a)$	spatial attenuation factor, equation (19)
T	temperature
W	vertical dependence of z-component of velocity
x	horizontal coordinate
z	vertical coordinate.

Greek symbols

δ	radiosity perturbation amplitude, equation (17)
ε	boundary or film spectral emissivity
ζ	horizontal variable of integration
θ	vertical dependence of temperature perturbation T', equation (10)
ν	photon wave number
ρ	boundary or film spectral reflectivity
σ	Stefan–Boltzmann constant
τ	film spectral transmissivity.

Subscripts

c	conductive–convective boundary condition
e	external to the fluid layer boundary
r	radiative boundary condition
ν	spectral radiation quantity
0	lower boundary at $z^* = 0$
1	upper boundary at $z^* = 1$
I	coordinate system in the lower fluid layer of the symmetric two fluid layer system
II	coordinate system in the upper fluid layer of the symmetric two fluid layer system.

Superscripts

*	dimensionless variable
$\bar{}$	mean quiescent quantity
$'$	perturbation quantity
+	radiant heat flux leaving a surface, radiosity
−	radiant heat flux falling on a surface, irradiation.

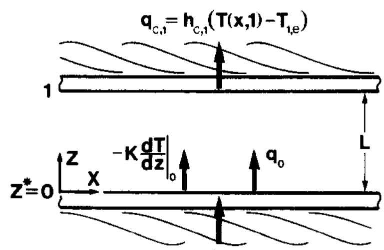

FIG. 1. The single layer.

Linear stability theory assumes small perturbations of temperature about the mean quiescent temperature profile

$$T(x,z) = \bar{T}(z) + T'(x,z) \tag{9}$$

and disturbances in the form

$$T'(x,z) = \theta(z)\cos ax^* \tag{10}$$

where a is the dimensionless wave number. Introducing equation (9) into equation (8), subtracting the mean heat balance equation, and arranging into the form of Sparrow *et al.* gives

$$-K\frac{\partial T'}{\partial z} = -h_{c,0}T' - q'_0 \tag{11}$$

where q'_0 remains to be evaluated.

For small temperature perturbations, the spectral black body radiosity B_ν may be expanded in terms of a base value $\bar{B}_\nu$ plus a perturbation B'_ν. For example, at $z = 0$ the x-variation in B_ν is

$$B_{\nu 0}(x) = B_\nu(\bar{T}(0)) + T'(x,0)(\partial B_\nu/\partial T)_{T=\bar{T}(0)} \tag{12}$$

where $B_\nu(T)$ is given by Planck's equation

$$B_\nu(T) = 2\pi hc^2\nu^3/(\exp(hc\nu/kT) - 1). \tag{13}$$

(For a 'gray' analysis, one would replace B_ν with σT^4 and $\partial B_\nu/\partial T$ with $4\sigma T^3$ in equation (12) and the equations that follow.) A radiosity–irradiation formulation may then be used for diffuse boundaries to relate net perturbation heat flux to $T'(x,0)$ and $T'(x,1)$. The spectral radiosity of the lower boundary is

$$q^+_{\nu 0}(x) = \varepsilon_0 B_{\nu 0}(x) + \rho_0 q^-_{\nu 0}(x) \tag{14}$$

where the irradiation is contributed by the shape-

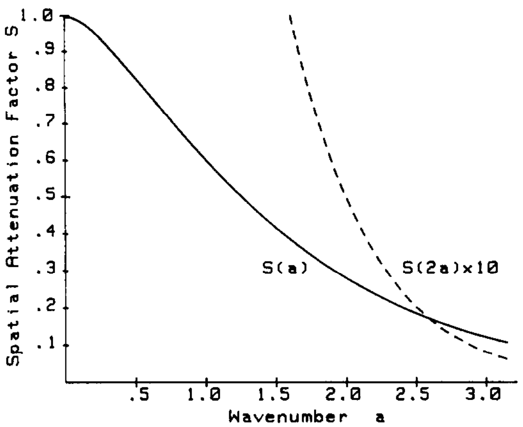

FIG. 2. The spatial attenuation factor $S(a)$.

factor-weighted average of the upper boundary radiosity

$$q_{v0}^{-}(x) = \int_{-\infty}^{\infty} q_{v1}^{+}(x+\zeta)\frac{dF_s}{d\zeta}d\zeta \tag{15}$$

where the shape-factor kernel is [7]

$$\frac{dF_s}{d\zeta} = \frac{L^2}{2[L^2+\zeta^2]^{3/2}}. \tag{16}$$

Similar relations hold for the upper boundary.

The perturbation term in equation (12) when introduced into equation (14) and the upper boundary relations introduced into equation (15) show that the radiosity and irradiation are composed of mean and perturbation quantities. The upper radiosity in the form of

$$q_{v1}^{+}(x) = \bar{q}_{v1}^{+} + \delta_{v1}\cos(ax^*) \tag{17}$$

gives rise to an irradiation on the lower boundary given by equation (15)

$$q_{v0}^{-}(x) = \bar{q}_1^{+} + \delta_{v1}S(a)\cos(ax^*) \tag{18}$$

where the spatial attenuation factor is

$$S(a) = \int_{-\infty}^{\infty} \cos(a\zeta^*)\frac{dF_s}{d\zeta^*}d\zeta^*, \quad \zeta^* = \zeta/L. \tag{19}$$

The perturbation irradiation is thus seen to be related to the perturbation radiosity by

$$q_{v0}^{-\prime}(x) = q_{v1}^{+\prime}(x)S(a). \tag{20}$$

It is evident that $S(a)$ shown in Fig. 2 plays the role of a geometric transmissivity that attenuates the radiant perturbation crossing the fluid layer. Note that all the radiation perturbation quantities are in phase with the temperature perturbation; all go as $\cos(ax^*)$.

The net perturbation radiant heat flux at the lower plate follows from the perturbation form of equations (14), (20), and the corresponding equations for the upper boundary, which can be obtained by interchanging subscripts 0 and 1. After integration over all photon wave numbers, there results

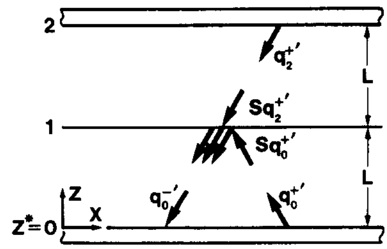

FIG. 3. Perturbation radiant fluxes in a twin layer.

$$q_0'(x) = F_0 h_{r,0} T'(x,0) - F_{01} h_{r,1} T'(x,1) \tag{21}$$

where the total transfer factors are

$$F_0 = \frac{1}{h_{r,0}}\int_0^{\infty} \frac{\varepsilon_0(1-\rho_1 S^2)}{1-\rho_0\rho_1 S^2}\left(\frac{\partial B_v}{\partial T}\right)_{T=\bar{T}(0)} dv \tag{22a}$$

$$F_{01} = \frac{1}{h_{r,1}}\int_0^{\infty} \frac{\varepsilon_0\varepsilon_1 S}{1-\rho_0\rho_1 S^2}\left(\frac{\partial B_v}{\partial T}\right)_{T=\bar{T}(1)} dv \tag{22b}$$

and h_r denotes the T-derivative of the total black body radiosity

$$h_{r,0} = 4\sigma\bar{T}^3(0), \quad h_{r,1} = 4\sigma\bar{T}^3(1). \tag{23a, b}$$

Equation (21) can be substituted into equation (11) and made dimensionless for comparison to the boundary condition of Sparrow *et al.* This and the upper boundary condition can be made to appear in the same form by introducing the ratio of the dimensionless temperature perturbation amplitudes of the boundaries

$$R_\theta = \theta^*(1)/\theta^*(0). \tag{24}$$

With this artifice, equation (11), the lower boundary condition, becomes

$$D_z\theta^* = Bi_0\theta^*, \quad z^* = 0 \tag{25}$$

where the new total Biot number is composed of three components

$$Bi_0 = Bi_{c,0} + F_0\,Bi_{r,0} - F_{01}\,Bi_{r,1}\,R_\theta \tag{26}$$

$$Bi_{c,0} = h_{c,0}L/K, \quad Bi_{r,0} = h_{r,0}L/K, \quad Bi_{r,1} = h_{r,1}L/K. \tag{27a–c}$$

Twin layers

The problem of two symmetric horizontal layers separated by a thin dielectric film can be treated by an extension of the single layer analysis. The outer boundaries must have the same total Biot numbers. For example, two layers of the same fluid and of equal thickness could have the same boundary emissivities and heat transfer coefficients, and the top and bottom surfaces of the film the same emissivity and reflectivity for symmetry. Subscripts 0, 1, and 2 are used to denote the lower boundary, film, and upper boundary, respectively. A specularly reflecting and transmitting film as shown in Fig. 3 is assumed, but the boundaries are again taken to be diffuse.

With the assumption of a specular film the radiosity irradiation equations can be written in terms of only

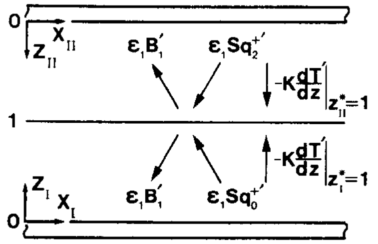

FIG. 4. Film perturbation heat balance for the twin layer.

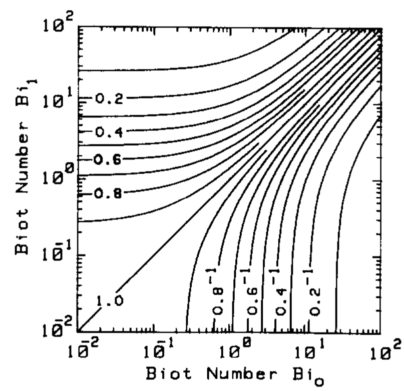

FIG. 5. Contours of constant amplitude ratio R_θ as a function of Bi_0 and Bi_1.

the outer boundaries. Proceeding as before the perturbation heat fluxes may be found. The perturbation heat flux condition at the lower boundary remains as equation (11). That for the film is

$$-K\frac{\partial T'}{\partial z}\bigg|_{z_I^*=1} - K\frac{\partial T'}{\partial z}\bigg|_{z_{II}^*=1} = q_1' \quad (28)$$

where the net perturbation spectral radiative flux from the film is the emission minus the absorption on each side as shown in Fig. 4. Note that ε_1 is a spectral quantity that is both emissivity and absorptivity

$$q_{v1}' = 2\varepsilon_1 B_{v1}' - \varepsilon_1 S(a) q_{v0}^{+\prime} - \varepsilon_1 S(a) q_{v2}^{+\prime}. \quad (29)$$

Introducing the boundary radiosities into equation (29) and integrating over all photon wave numbers gives the net perturbation total radiative flux in the form

$$q_1' = F_1 h_{r,1} T'(x,1) - F_{10} h_{r,0} T'(x,0) - F_{12} h_{r,2} T'(x,2). \quad (30)$$

The total transfer factors are integrals of the spectral ones

$$h_{r,1} F_1 = \int_0^\infty F_{v1} (\partial B_v/\partial T)_{T=\bar{T}(1)}\, dv \quad (31a)$$

$$h_{r,0} F_{10} = \int_0^\infty F_{v10} (\partial B_v/\partial T)_{T=\bar{T}(0)}\, dv \quad (31b)$$

$$h_{r,2} F_{12} = \int_0^\infty F_{v12} (\partial B_v/\partial T)_{T=\bar{T}(2)}\, dv \quad (31c)$$

and the spectral transfer factors are

$$F_{v1} = 2\varepsilon_1 - \frac{\varepsilon_1^2 S(a)^2}{D_{02}} [\rho_0 + \rho_2 + 2\rho_0\rho_2(\tau_1 - \rho_1) S(2a)] \quad (32a)$$

$$F_{v10} = F_{v01} = \frac{1}{D_{02}} [1 + \rho_2(\tau_1 - \rho_1) S(2a)] \varepsilon_0 \varepsilon_1 S(a) \quad (32b)$$

$$F_{v12} = F_{v21} = \frac{1}{D_{02}} [1 + \rho_0(\tau_1 - \rho_1) S(2a)] \varepsilon_2 \varepsilon_1 S(a) \quad (32c)$$

where

$$D_{02} = [1 - \rho_0\rho_1 S(2a)][1 - \rho_2\rho_1 S(2a)] - \rho_0\rho_2\tau_1^2 S^2(2a). \quad (32d)$$

The perturbation radiant flux on the lower boundary is

$$q_0' = F_0 h_{r,0} T'(x,0) - F_{01} h_{r,1} T'(x,1) - F_{02} h_{r,2} T'(x,2) \quad (33)$$

where the new total transfer factors in equation (33) result from spectral integration of

$$F_{v0} = \frac{\varepsilon_0}{D_{02}} [1 - \rho_2\rho_1 S(2a)] \quad (34a)$$

$$F_{v02} = \frac{\varepsilon_0 \varepsilon_2 \tau S(2a)}{D_{02}} \quad (34b)$$

as in equations (31a)–(31c). Note that the plot of $S(a)$ is extended to $2a = 6.2$ in Fig. 2 to accommodate values of a as large as 3.1.

To reduce the symmetric two-layer problem to the one-layer problem, a new coordinate system is introduced as shown in Fig. 4

$$z_I^* = z_I/L \quad (35a)$$

$$z_{II}^* = z_{II}/L, \quad z_{II} = 2L - z. \quad (35b)$$

In this coordinate system the governing equations and boundary conditions for the upper layer are identical to those written in terms of z_I^* for the lower layer. Accordingly the solution for thermal stability of each layer is the same as for the single layer with new Biot numbers defined from the introduction of equations (30) and (33) into the perturbation heat balances, equations (11) and (28). The modified Biot numbers are

$$Bi_0 = Bi_{c,0} + (F_0 - F_{02}) Bi_{r,0} - F_{01}\, Bi_{r,1}\, R_\theta \quad (36)$$

$$Bi_1 = \tfrac{1}{2}[F_1\, Bi_{r,1} - (F_{10} + F_{12}) Bi_{r,0}/R_\theta]. \quad (37)$$

RESULTS

Because of the need for amplitude ratio R_θ in the modified Biot numbers, and the need for arbitrary Biot numbers simultaneously on each boundary, it was necessary to repeat and extend the calculations of Sparrow *et al.* Figure 5 shows contours of constant

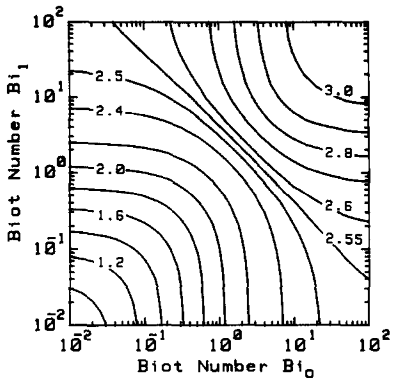

FIG. 6. Contours of constant critical wave number as a function of Bi_0 and Bi_1.

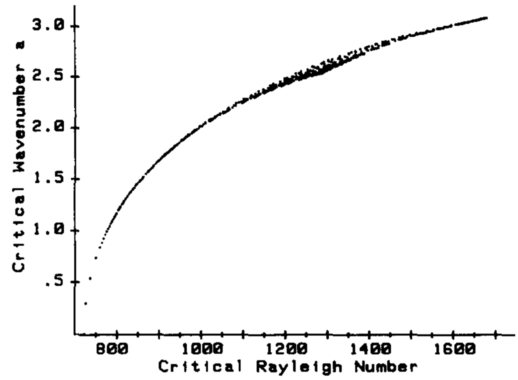

FIG. 8. Critical wave number vs critical Rayleigh number.

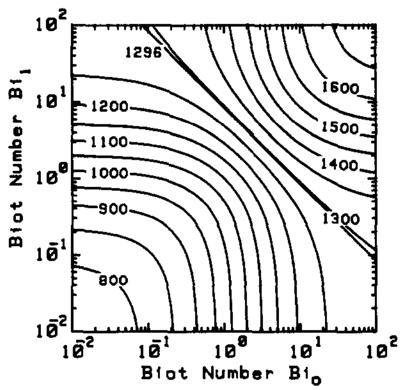

FIG. 7. Contours of constant critical Rayleigh number as a function of Bi_0 and Bi_1.

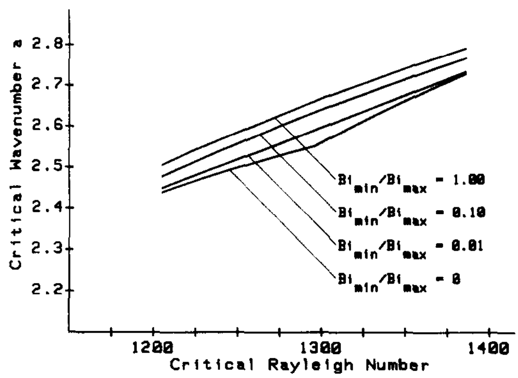

FIG. 9. Critical wave number vs critical Rayleigh number for selected Biot number ratios.

R_θ as a function of Bi_0 and Bi_1. The wave number a which minimizes the critical Rayleigh number is also needed to calculate $S(a)$ and $S(2a)$ in the transfer factors which contribute to the Biot numbers. Figure 6 shows contours of critical wave number vs Biot numbers. Clearly, for practical calculations, an iterative procedure must be followed. With trial values of say $R_\theta = 1$ and $a = 3$, first values of Bi_0 and Bi_1 are calculated, and these values are used to enter Figs. 5 and 6 to improve the trial values, and refined values of Bi_0 and Bi_1 are found. It then remains to find Ra vs Bi_0 and Bi_1. For this purpose Fig. 7 is convenient.

Note that Figs. 5–7 are symmetric about the line $Bi_0 = Bi_1$. On Fig. 7 the contour for $Ra = 1296$ is shown because it is the value for $Bi_0 = 0$ and $Bi_1 = \infty$ or vice versa.

DISCUSSION

The similarity displayed by Figs. 6 and 7 suggested a plot of critical Rayleigh number vs critical wave number. Since Ra and a are functions of Bi_0 and Bi_1, a cross-plot of Ra vs a is a function of one Biot number or the ratio Bi_{min}/Bi_{max}. Figure 8 shows points plotted without regard for the ratio. The plot therefore appears as a band, but the width of the band is narrow. As both Bi_0 and Bi_1 both approach zero, wave number goes to zero at $Ra = 720$, as shown by Sparrow *et al.* As Bi_0 and Bi_1 both approach infinity, wave number goes to the well-known result $a = 3.117$ at $Ra = 1708$. At either extremity of Fig. 8 the band is quite narrow. Only in the vicinity of $Ra = 1296$ shown magnified in Fig. 9 is there a spread from approximately $a = 2.55$ to 2.65 as B_{min}/B_{max} goes from 0 to 1, a variation of only 4%. Thus for practical calculations Fig. 8 may be used in place of Fig. 6 or 7.

Some feeling for the practical import of the present results may be gained from a short example. Consider a non-selective black solar collector with the absorber plate and coverglass approximated as black and at prescribed temperatures (h_0 and h_2 large) separated by two 10 mm air gaps about a Teflon film, approximated as having a total emissivity of 0.28 and infrared reflectivity and transmissivity of 0.08 and 0.64, respectively. If one neglected the radiant interaction, one would have $Bi_0 = \infty$ and $Bi_1 = 0$ to give $Ra = 1296$. Considering the radiant interaction (with properties evaluated at 325 K) gives from equations (36) and (37), $Bi_1 = 0.77$, and $Bi_0 = \infty$. Figure 7 then indicates Ra is approximately 1380.

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REFERENCES

1. E. M. Sparrow, R. J. Goldstein and V. K. Jonsson, Thermal instability in a horizontal fluid layer: effect of boundary conditions and nonlinear temperature profile, *J. Fluid Mech.* **18**, 513–528 (1964).
2. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Chap. II. Clarendon Press, Oxford (1961).
3. D. K. Edwards and W. M. Sun, Effect of wall radiation on thermal instability in a vertical cylinder, *Int. J. Heat Mass Transfer* **14**, 15–18 (1971).
4. I. Catton and J. H. Lienhard V, Thermal stability of two fluid layers separated by a solid interlayer of finite thickness and thermal conductivity, *ASME J. Heat Transfer* **106**, 605–612 (1984).
5. J. H. Lienhard V and I. Catton, Heat transfer across a two-fluid-layer region, *ASME J. Heat Transfer* **108**, 198–205 (1986).
6. J. H. Lienhard V, An improved approach to conductive boundary conditions for the Rayleigh–Benard instability, *ASME J. Heat Transfer* **109**, 378–387 (1987).
7. E. M. Sparrow and R. D. Cess, *Radiation Heat Transfer*, Chap. 3. Hemisphere/McGraw-Hill, Washington, DC (1978).

EFFET DU RAYONNEMENT DES FRONTIERES SUR LA STABILITE THERMIQUE DES COUCHES HORIZONTALES

Résumé—Le rayonnement thermique à travers des couches horizontales de fluides neutres vis-à-vis du rayonnement, augmente le nombre de Rayleigh critique en modifiant les conditions aux limites thermiques. Une modification de condition par le rayonnement peut être caractérisée par deux nombres de Biot modifiés qui incluent la contribution radiative. On présente les rapports d'amplitudes de perturbations nécessaires pour calculer ces nombres. Le nombre d'onde de la perturbation la plus instable dépend principalement du nombre de Rayleigh critique et seulement faiblement du rapport des nombres de Biot modifés, ce qui facilite les applications pratiques.

DER EINFLUSS DER WÄRMESTRAHLUNG VON BEGRENZUNGSWÄNDEN AUF DIE STABILITÄT VON WAAGERECHTEN SCHICHTEN

Zusammenfassung—Wärmestrahlung durch die horizontalen Schichten eines Fluids, das am Strahlungsaustausch nicht teilnimmt, erhöht die kritische Rayleigh-Zahl über eine Änderung der thermischen Randbedingungen. Die Änderung der konvektiven Randbedingung durch Strahlung kann durch zwei modifizierte Biot-Zahlen beschrieben werden, die einen Strahlungsanteil beinhalten. Die Amplitudenverhältnisse der Störungen, die für die Berechnung der modifizierten Biot-Zahl nötig sind, werden vorgestellt. Die Wellenzahl der am wenigsten stabilen Störung hängt vor allem von der kritischen Rayleigh-Zahl und nur wenig vom Verhältnis der modifizierten Biot-Źahlen ab und erleichtert so die Anwendung in der Praxis.

ВЛИЯНИЕ ИЗЛУЧЕНИЯ НА ГРАНИЦЕ НА ТЕПЛОВУЮ УСТОЙЧИВОСТЬ В ГОРИЗОНТАЛЬНЫХ СЛОЯХ

Аннотация—Радиационный теплоперенос в неизлучающих горизонтальных слоях жидкости, меняя тепловые граничные условия, увеличивает критическое число Рэлея. Влияние излучения на граничные условия для конвекции можно охарактеризовать двумя модифицированными числами Био, учитывающими вклад излучения. Представлены соотношения для расчета модифицированных чисел Био. Показано, что волновое число наиболее неустойчивого возмущения главным образом зависит от критического числа Рэлея и слабо от соотношения модифицированных чисел био, что облегчает техническое применение этих выводов.